Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

ON FUNCTIONAL DETERMINANTS

BY PAUL SAUREL

THE following theorem is well known:[*] If n functions $u_1, u_2, \cdots, u_n$ of the n variables $x_1, x_2, \cdots, x_n$ are connected by a relation

$$f(u_1, u_2, \cdots, u_n) = 0,$$

the functional determinant

$$\frac{\partial(u_1, u_2, \cdots, u_n)}{\partial(x_1, x_2, \cdots, x_n)}$$

vanishes; and, conversely, if the functional determinant vanishes, there exists a relation connecting the n functions.

In the present note we shall give a demonstration of the following more general but less known theorem:[†] If r functions $u_1, u_2, \cdots, u_r$ of the n variables $x_1, x_2, \cdots, x_n$, r being less than or equal to n, are connected by a relation

$$f(u_1, u_2, \cdots, u_r) = 0, \tag{1}$$

every determinant of order r formed from the matrix

$$\begin{matrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdot\;\cdot\;\cdot\;\cdot\;\cdot & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdot\;\cdot\;\cdot\;\cdot\;\cdot & \frac{\partial u_2}{\partial x_n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{\partial u_r}{\partial x_1} & \frac{\partial u_r}{\partial x_2} & \cdot\;\cdot\;\cdot\;\cdot\;\cdot & \frac{\partial u_r}{\partial x_n} \end{matrix} \tag{2}$$

vanishes; and, conversely, if every determinant of order r formed from the above matrix vanishes while in each of the r matrices which can be derived from the given matrix by suppressing one row there is at least one determinant of order $r - 1$ which does not vanish, there exists a relation connecting the r functions and involving all of them.

---

[*] C. G. J. Jacobi, *Journal für die reine und angewandte Mathematik*, vol. 22, p. 328; 1841. Ostwald's *Klassiker der exacten Wissenschaften*, No. 78, Ueber die Functionaldeterminanten, p. 16.

[†] Loc. cit., p. 347 or p. 42.

The demonstration of the first part of the theorem is very simple. If equation 1 holds, then the n equations

$$\sum_{i=1}^{r} \frac{\partial f}{\partial u_i} \frac{\partial u_i}{\partial x_j} = 0 \qquad (j = 1, 2, \cdots, n) \tag{3}$$

obtained by differentiating equation 1 will also hold. Now, since each of the r functions is supposed to appear in the given relation 1, it follows that no one of the expressions $\frac{\partial f}{\partial u_i}$ is identically equal to zero; it then follows at once from equations 3 that every determinant of order r formed from the matrix 2 must be equal to zero.

Let us now suppose that every determinant of order r formed from the matrix 2 is equal to zero and that in each of the matrices obtained by suppressing one row of the given matrix there is at least one determinant of order $r - 1$ which is not equal to zero. Since in the matrix which is formed by suppressing the last row in (2) there is at least one determinant of order $r - 1$ which is not equal to zero, we may assume, without loss of generality, that

$$\frac{\partial(u_1, u_2, \cdots, u_{r-1})}{\partial(x_1, x_2, \cdots, x_{r-1})} \neq 0. \tag{4}$$

From this it follows that at least one of the expressions $\frac{\partial u_1}{\partial x_1}, \frac{\partial u_1}{\partial x_2}, \cdots, \frac{\partial u_1}{\partial x_{r-1}}$ is not equal to zero; we may therefore assume, again without loss of generality, that

$$\frac{\partial u_1}{\partial x_1} \neq 0. \tag{5}$$

Inequality 5 shows that u_1 depends upon x_1; it is therefore possible to express x_1 as a function of $u_1, x_2, x_3, \cdots, x_n$. Using this value of x_1 we can then express $u_2, u_3, \cdots, u_r$ as functions of $u_1, x_2, x_3, \cdots, x_n$. If we denote the resulting expressions by $u_2', u_3', \cdots, u_r'$ we get without difficulty

$$\begin{aligned} \frac{\partial u_i}{\partial x_1} &= \frac{\partial u_i'}{\partial u_1}\frac{\partial u_1}{\partial x_1} && (i = 2, 3, \cdots, r), \\ \frac{\partial u_i}{\partial x_j} &= \frac{\partial u_i'}{\partial u_1}\frac{\partial u_1}{\partial x_j} + \frac{\partial u_i'}{\partial x_j} && (j = 2, 3, \cdots, n). \end{aligned} \tag{6}$$

If we substitute these values in (4) and change the form of the resulting determinant by subtracting from the elements of the i^{th} row those of the first row multiplied by $\frac{\partial u_i'}{\partial u_1}$, we find that

$$\frac{\partial u_1}{\partial x_1} \cdot \frac{\partial(u_2', u_3', \cdots, u_{r-1}')}{\partial(x_2, x_3, \cdots, x_{r-1})} \neq 0. \tag{7}$$

From (5) and (7) it follows that

$$\frac{\partial(u_2', u_3', \cdots, u_{r-1}')}{\partial(x_2, x_3, \cdots, x_{r-1})} \neq 0. \tag{8}$$

In like manner formulas 5 and 6 will enable us to replace the equations

$$\frac{\partial(u_1, u_2, \cdots, u_{r-1}, u_r)}{\partial(x_1, x_2, \cdots, x_{r-1}, x_{r+s})} = 0 \qquad (s = 0, 1, 2, \cdots, n-r), \tag{9}$$

by the equivalent equations

$$\frac{\partial(u_2', u_3', \cdots, u_{r-1}', u_r')}{\partial(x_2, x_3, \cdots, x_{r-1}, x_{r+s})} = 0 \qquad (s = 0, 1, 2, \cdots, n-r). \tag{10}$$

If we remember that when equations 9 hold every determinant of order r formed from the matrix 2 vanishes, it follows from (10) that every determinant of order $r - 1$ formed from the matrix

$$\begin{matrix} \frac{\partial u_2'}{\partial x_2} & \frac{\partial u_2'}{\partial x_3} & \cdots\cdots & \frac{\partial u_2'}{\partial x_n} \\ \frac{\partial u_3'}{\partial x_2} & \frac{\partial u_3'}{\partial x_3} & \cdots\cdots & \frac{\partial u_3'}{\partial x_n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{\partial u_r'}{\partial x_2} & \frac{\partial u_r'}{\partial x_3} & \cdots\cdots & \frac{\partial u_r'}{\partial x_n} \end{matrix} \tag{11}$$

vanishes.

Starting now with the matrix 11 and the inequality 8, and repeating the reasoning which we applied to the matrix 2 and the inequality 4, we can show that

$$\frac{\partial(u_3'', u_4'', \cdots, u_{r-1}'')}{\partial(x_3, x_4, \cdots, x_{r-1})} \neq 0, \tag{12}$$

and that every determinant of order $r-2$ formed from the matrix

$$\begin{matrix} \frac{\partial u_3''}{\partial x_3} & \frac{\partial u_3''}{\partial x_4} & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \frac{\partial u_3''}{\partial x_n} \\ \frac{\partial u_4''}{\partial x_3} & \frac{\partial u_4''}{\partial x_4} & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \frac{\partial u_4''}{\partial x_n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{\partial u_r''}{\partial x_3} & \frac{\partial u_r''}{\partial x_4} & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \frac{\partial u_r''}{\partial x_n} \end{matrix} \tag{13}$$

vanishes. Here $u_3'', u_4'', \cdots, u_r''$ denote the functions of $u_1, u_2', x_3, \cdots, x_n$ which are obtained from $u_3', u_4', \cdots, u_r'$ by replacing x_2 by its value in terms of $u_1, u_2', x_3, \cdots, x_n$.

By repeating this process we shall finally arrive at the result that every term in the matrix

$$\frac{\partial u_r^{(r-1)}}{\partial x_r} \quad \frac{\partial u_r^{(r-1)}}{\partial x_{r+1}} \cdots \frac{\partial u_r^{(r-1)}}{\partial x_n} \tag{14}$$

must be equal to zero. As $u_r^{(r-1)}$ denotes the expression which is obtained when u_r is expressed in terms of $u_1, u_2', u_3'', \cdots, u_{r-1}^{(r-2)}, x_r, x_{r+1}, \cdots, x_n$, it follows that $u_r^{(r-1)}$ is a function of $u_1, u_2', u_3'', \cdots, u_{r-1}^{(r-2)}$ alone. Dropping the primes we may therefore say that there exists a relation connecting the r functions $u_1, u_2, \cdots, u_r$. Moreover this relation must involve each of the r functions. For, if one of them were missing from it, every determinant of order $r-1$ formed from the matrix corresponding to the remaining $r-1$ functions would vanish; but this contradicts the hypothesis. The proposition is thus established.

New York, December, 1905.